Derivations

$\mathcal{A}$ an $\mathbb{F}-$algebra (not necessary associative) with product $\Diamond$

$$\mathcal{A} \times \mathcal{A} \ni (a, b) \longrightarrow a \Diamond b \in \mathcal{A}$$

$\Diamond$ bilinear mapping, ∂ is a derivation of $\mathcal{A}$ if ∂ is a linear map $\mathcal{A} \xrightarrow{\partial} \mathcal{A}$ satisfying the Leibnitz property:

$$\partial\left(A \Diamond B\right) = A \Diamond \left(\partial(B)\right) + \left(\partial(A)\right) \Diamond B$$

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$\mathfrak{Der}\left(\mathcal{A}\right) =$ all derivations on $\mathcal{A}$

Prop: $\mathfrak{Der}\left(\mathcal{A}\right)$ is a Lie Algebra $\subset \mathfrak{gl}(\mathcal{A})$ with commutator

$$[\partial, \partial'] (A) \equiv \partial\left(\partial'(A)\right) - \partial'\left(\partial(A)\right)$$

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$\boxed{\text{Der}(\mathcal{A}) \text{ is a Lie Algebra}}$

$\mathcal{A}$ άλγεβρα = γραμμικός χώρος χώρος με γινόμενο $\bullet\diamond\bullet$, $\text{Der}(\mathcal{A}) \subset gl(\mathcal{A}) = \text{End}(\mathcal{A})$

$\partial(B\diamond C) = (\partial B)\diamond C + B\diamond(\partial C)$ δηλαδή ισχύει ο κανόνας Leibnitz.

Απόδειξη: Αν $\partial_1,\partial_2 \in \text{Der}(\mathcal{A}) \rightsquigarrow [\partial_1,\partial_2] = \partial_1\circ\partial_2 - \partial_2\circ\partial_1 \in \text{Der}(\mathcal{A})$. διότι

$$[\partial_1,\partial_2](B\diamond C) = \partial_1(\partial_2(B\diamond C)) - \partial_2(\partial_1(B\diamond C)) =$$

$$\partial_1(\partial_2 B\diamond C + B\diamond\partial_2 C) - \partial_2(\partial_1 B\diamond C + B\diamond C) =$$

$$\partial_1(\partial_2 B)\diamond C + \cancel{(\partial_2 B)\diamond(\partial_1 C)} + \cancel{(\partial_1 B)\diamond(\partial_2 C)} + B\diamond\partial_1(\partial_2 C) -$$

$$-\partial_2(\partial_1 B)\diamond C - \cancel{(\partial_1 B)\diamond(\partial_2 C)} - \cancel{(\partial_2 B)\diamond(\partial_1 C)} - B\diamond\partial_2(\partial_1 B) =$$

$$= [\partial_1,\partial_2](B)\diamond C + B\diamond[\partial_1,\partial_2](C) \rightsquigarrow [\partial_1,\partial_2] \in \text{Der}(\mathcal{A})$$ δηλαδή

$$[\text{Der}(\mathcal{A}), \text{Der}(\mathcal{A})] \subset \text{Der}(\mathcal{A}) \Longrightarrow \boxed{\text{Der}\mathcal{A} \text{ Lie subalgebra of } gl(\mathcal{A}).}$$

Derivations on a Lie algebra

$\mathfrak{g}$ a $\mathbb{F}-$ Lie algebra, ∂ is a Lie derivation of $\mathfrak{g}$ if ∂ is a linear map:

$$\mathfrak{g} \ni x \xrightarrow{\partial} \partial(x) \in \mathfrak{g}$$

satisfying the Leibnitz property: $\partial([x, y]) = [\partial(x), y] + [x, \partial(y)]$

$\mathfrak{Der}(\mathfrak{g})$ = all derivations on $\mathfrak{g}$

Prop: $\mathfrak{Der}(\mathfrak{g})$ is a Lie Algebra $\subset \mathfrak{gl}(\mathfrak{g})$ with commutator
$$[\partial, \partial'](x) \equiv \partial(\partial'(x)) - \partial'(\partial(x))$$

Αμεση εφαρμογή της ασκήσεως 12

Prop: $\delta \in \mathfrak{Der}(\mathfrak{g}) \rightsquigarrow \delta^n([x, y]) = \sum_{k=0}^{n} \binom{n}{k} \left[\delta^k(x), \delta^{n-k}(y)\right]$

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Prop: $\delta \in \mathfrak{Der}(\mathfrak{g}) \rightsquigarrow \mathrm{e}^{\delta}([x, y]) = \left[\mathrm{e}^{\delta}(x), \mathrm{e}^{\delta}(y)\right] \leftrightsquigarrow \mathrm{e}^{\delta} \in \mathrm{Aut}(\mathfrak{g})$

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Proposition

$\phi(t)$ smooth monoparametric familly in $\mathrm{Aut}(\mathfrak{g})$ and $\phi(0) = \mathrm{Id} \rightsquigarrow$
$\phi'(0) \in \mathfrak{Der}(\mathfrak{g})$

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Adjoint Representation

Definition (Adjoint Representation)

Let $x \in \mathfrak{g}$ and $\mathrm{ad}_x : \mathfrak{g} \longrightarrow \mathfrak{g}$: $\mathrm{ad}_x(y) = [x, y]$

Prop: $\boxed{\mathrm{ad}_{[x,y]} = [\mathrm{ad}_x, \mathrm{ad}_y]}$ $\Rightarrow$ $\mathrm{ad}_x \in \mathfrak{Der}(\mathfrak{g})$

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Prop: $\boxed{\mathrm{ad}_{\mathfrak{g}} \equiv \bigcup_{x \in \mathfrak{g}} \{\mathrm{ad}_x\} \text{ Lie-subalgebra of } \mathfrak{Der}(\mathfrak{g})}$

Άμεση συνέπεια της παραπάνω πρότασης

$[\mathrm{ad}_{\mathfrak{g}}, \mathrm{ad}_{\mathfrak{g}}] \subseteq \mathrm{ad}_{\mathfrak{g}}$.

Definition: **Inner Derivations**$=\mathrm{ad}_{\mathfrak{g}}$

Definition **Outer Derivations**$= \mathfrak{Der}(\mathfrak{g}) \setminus \mathrm{ad}_{\mathfrak{g}}$

Prop: $$\boxed{\left\{ \begin{array}{c} x \in \mathfrak{g} \\ \delta \in \mathfrak{Der}(\mathfrak{g}) \end{array} \right\} \rightsquigarrow \left\{ [\mathrm{ad}_x, \delta] = -\mathrm{ad}_{\delta(x)} \right\} \rightsquigarrow \left\{ \begin{array}{c} \mathrm{ad}_{\mathfrak{g}} \\ \text{ideal of} \\ \mathfrak{Der}(\mathfrak{g}) \end{array} \right\}}$$

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Prop:

$$\boxed{\tau \in \mathrm{Aut}(\mathfrak{g}) \leftrightsquigarrow \tau([x, y]) = [\tau(x), \tau(y)] \rightsquigarrow \mathrm{ad}_{\tau(x)} = \tau \circ \mathrm{ad}_x \circ \tau^{-1}}$$

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$\boxed{\forall x,y \in \mathfrak{g},\ ad_x y \overset{\text{op}}{\equiv} [x,y]} \Rightarrow \boxed{ad_{[x,y]} = [ad_x, ad_y]} \Rightarrow \boxed{ad_x \in \mathcal{D}er\,\mathfrak{g}}$

Απόδειξη: αν $x,y,z \in \mathfrak{g} \rightsquigarrow ad_{[x,y]} z = [[x,y],z] \underset{\text{νόμος Leibnitz}}{=} \underbrace{[[x,z],y]}_{-[y,[x,z]] = -ad_y([x,z])} + \underbrace{[x,[y,z]]}_{ad_x([y,z])} = -ad_y([x,z]) + ad_x([y,z])$

$$ad_{[x,y]} z = -ad_y(ad_x z) + ad_x(ad_y z)$$

$$\Rightarrow ad_{[x,y]} z = (ad_x \circ ad_y) z - (ad_y \circ ad_x) z = [ad_x, ad_y] z \Rightarrow \boxed{ad_{[x,y]} = [ad_x, ad_y]}$$

$$\Rightarrow ad_x \in \mathcal{D}er[\mathfrak{g}].$$

Matrix Form of the adjoint representation

$$\mathfrak{g} = \mathrm{span}\left(e_1, e_2, \dots, e_n\right) = \mathbb{C}e_1 + \mathbb{C}e_2 + \cdots + \mathbb{C}e_n$$

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k\, e_k, \qquad c_{ij}^k \longleftrightarrow \text{structure constants}$$

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Antisymmetry: $c_{ij}^k = -c_{ji}^k$

Jacobi identity $c_{i\,j}^{m} c_{m\,k}^{\ell} + c_{j\,k}^{m} c_{m\,i}^{\ell} + c_{k\,i}^{m} c_{m\,j}^{\ell} = 0$

$$e^\ell = e_\ell^* \in \mathfrak{g}^* \rightsquigarrow e_\ell^*.e_p = e^\ell.e_p = \delta_p^\ell$$

$$\mathbb{P}_i \in \mathfrak{gl}(\mathfrak{g}) \;:\; \boxed{e^k.\mathbb{P}_i e_j = (\mathbb{P}_i)_j^k = c_{ij}^k} \rightsquigarrow$$

$$[\mathbb{P}_i, \mathbb{P}_j] = \sum_{k=1}^{n} c_{ij}^k\, \mathbb{P}_k \rightsquigarrow$$

$$\boxed{\mathbb{P}_i = \mathrm{ad}_{e_i}}$$

Representations, Modules

V $\mathbb{F}$-vector space, $u, v, \ldots \in V$, $\alpha, \beta, \ldots \in \mathbb{F}$

Definition

Representation (ρ, V)

$$\mathfrak{g} \ni x \xrightarrow{\rho} \rho(x) \in \mathfrak{gl}(V)$$
$$V \ni v \xrightarrow{\rho(x)} \rho(x)v \in V$$
$$\rho(x) \text{ Lie homomorphism}$$

$$\begin{aligned} \rho(\alpha\, x + \beta\, y) &= \alpha\, \rho(x) + \beta\, \rho(y) \\ \rho\left([x, y]\right) &= [\rho(x), \rho(y)] = \\ &= \rho(x) \circ \rho(y) - \rho(y) \circ \rho(x) \end{aligned}$$

$V = \mathbb{C}^n \ \rightsquigarrow\ \rho(x) \in M_n(\mathbb{C}) = \mathfrak{gl}(V) = \mathfrak{gl}\left(\mathbb{C}, n\right)$

$W \subset V$ invariant (stable) subspace

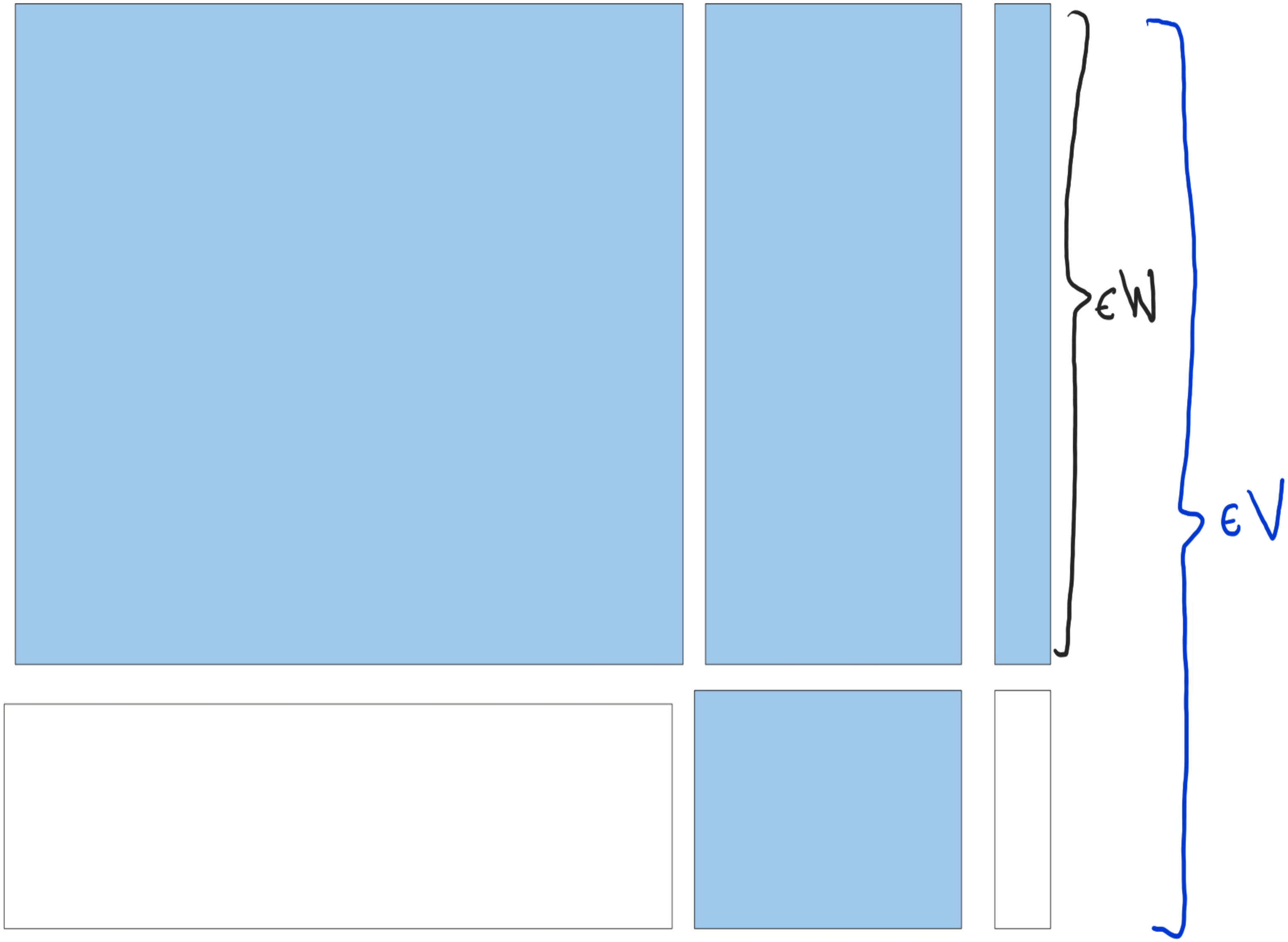
$\leftrightsquigarrow$ $\rho(\mathfrak{g})W \subset W$

$\leftrightsquigarrow$ (ρ, W) is submodule $\leftrightsquigarrow$ $(\rho, W) \prec (\rho, V)$

Το σύνολο των αναπαραστάσεων είναι μερικά διατεταγμένο.

$\forall x \in \mathfrak{g}$

$\rho(x) =$



$\rho(x) W \subset W$

Theorem

W invariant subspace of V $\leftrightsquigarrow (\rho, W) \prec (\rho, V)$

$(\rho, V) \rightsquigarrow \exists!$ $\boxed{\textit{induced representation}}$ $(\overline{\rho}, V/W)$ (quotient module)

$$\rho(\mathfrak{g})W \subset W \implies \exists!\, \overline{\rho}(x) : \; V/W \longrightarrow V/W$$

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$$\implies \begin{tikzcd} V \arrow[r, "\pi"] \arrow[d, "\rho(x)"'] \arrow[dr, dotted] & V/W \arrow[d, dashed, "\overline{\rho}(x)"] \\ V \arrow[r, "\pi"] & V/W \end{tikzcd} \quad \leftrightsquigarrow \quad \overline{\rho}(x) \circ \pi = \pi \circ \rho(x)$$

$\operatorname{Ker} \rho = C_\rho(\mathfrak{g}) = \{x \in \mathfrak{g} : \; \rho(x)V = \{0\}\}$ is an ideal

$\operatorname{Ker} \operatorname{ad} = C_{\operatorname{ad}}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g}) =$ center of $\mathfrak{g}$

$$\mathfrak{g} \xrightarrow[\text{Lie-epi}]{\rho} \rho(\mathfrak{g})$$

$x \in \operatorname{Ker}\rho \rightsquigarrow \rho(x)V = \{0\} \rightsquigarrow x \in C_\rho(\mathfrak{g})$

$x \in C_\rho(\mathfrak{g}) \rightsquigarrow \rho(x)V = \{0\} \rightsquigarrow x \in \operatorname{Ker}\rho$.

Συμ. $\boxed{\rho(C_\rho) = 0}$

Αν $\rho = \operatorname{ad}$

$\rightsquigarrow C_{\operatorname{ad}}(\mathfrak{g}) = \mathcal{Z}(\mathfrak{g})$

δηλ. $\operatorname{ad}_{C_{\operatorname{ad}}} = 0$.

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Αν $(\rho, W) \propto (\rho, V)$ τότε υπάρχει $(\bar{\rho}, V/W)$ έτσι ώστε $\forall x \in G$ το διάγραμμα είναι κλειστό

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[d, "\rho(x)"'] & V/W \arrow[d, "\bar{\rho}(x)"] \\ V \arrow[r, "\pi"] & V/W \end{tikzcd} \qquad \text{δλδ} \quad \bar{\rho}(x)\circ\pi = \pi\circ\rho(x) \qquad (*)$$

Απόδειξη.

$$\begin{tikzcd} V \arrow[d, "\rho(x)"'] \arrow[dr, "\pi\circ\rho(x)"] & \\ V \arrow[r, "\pi"', "\text{epi}"] & V/W \end{tikzcd}$$

Αν $v \in W \rightsquigarrow \rho(x)v \in W \rightsquigarrow \pi(\rho(x)v) = 0 \rightsquigarrow$
$v \in \mathrm{Ker}(\pi\circ\rho(x)) \rightsquigarrow W \subseteq \mathrm{Ker}(\pi\circ\rho(x))$

$$\begin{tikzcd} V \arrow[r, "\pi", "\text{epi}"'] \arrow[dr, "\pi\circ\rho(x)"'] & V/W \arrow[d, "\exists!\,\bar{\rho}(x)"] \\ & V/W \end{tikzcd}$$

$$\left.\begin{array}{l} \mathrm{Ker}\,\pi = W \subseteq \mathrm{Ker}(\pi\circ\rho) \\ \pi \text{ epi} \end{array}\right\} \Rightarrow \exists!\, \bar{\rho}(x)$$

και το διάγραμμα $(*)$ είναι κλειστό

δλδ $\bar{\rho}(x)\circ\pi = \pi\circ\rho(x)$